Title: Schwarz–type Boundary Value Problems for Inhomogeneous Cauchy-Riemann equation in Sector Planer Domains

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Abstract

This article aims to discuss the Schwarz boundary value problem for inhomogeneous Cauchy-Riemann equation in bounded and unbounded sectors in the complex plane.

Firstly, we discuss the Schwarz-Poisson formula in the sectors with angle $\pi/\alpha\ (\alpha \geq 1/2)$ by proper conformal mappings.

Secondly, the boundary behaviors of some linear integrals will be discussed, especially at the corner points.

Finally, the solutions and the conditions of solvability of the Schwarz problem for inhomogeneous Cauchy-Riemann equation in the sectors are obtained.

Introduction / Review

Complex analysis is a comparatively active branch in mathematics which has grown significantly. In particular, the investigation of boundary value problems for analytic and generalized analytic functions possesses both theoretical and applicable values of importance to many fields, such as electricity and magnetism, hydrodynamics, quantum mechanics, fluid dynamics, and so on [1-5].

The basic boundary value problems in complex analysis are Schwarz, Dirichlet, Neumann, and Robin problems. Kinds of boundary value problems have been investigated for various complex partial differential equations in some special domains[6-31].

Function Theoretical Main Tools

The main tools for solving boundary value problems of complex differential equations are the Gauss theorem, the Cauchy–Pompeiu representation, and Schwarz-Poisson formula.

❖ **Gauss theorem[6]:** *Let $D \subset \mathbb{C}$ be a regular domain.* $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$, $\quad z = x + iy$, *then*

$$\int_D w_{\bar z}(z)\,dxdy = \frac{1}{2i}\int_{\partial D} w(z)dz, \quad \int_D w_z(z)\,dxdy = -\frac{1}{2i}\int_{\partial D} w(z)d\bar z$$

❖ **Cauchy–Pompeiu formula [6]:** *Any* $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$ *for a regular complex domain* $D \subset \mathbb{C}$ *can be represented as*

$$\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - z} = \begin{cases} w(z), & z \in D, \\ 0, & z \notin \overline{D}, \end{cases}$$

❖ **Schwarz-Poisson formula in the unit disc [6]:** *Let D be the unit disc , if* $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$, *then*

$$w(z) = \frac{1}{2\pi i}\int_{|\zeta|=1} \operatorname{Re} w(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} + \frac{1}{2\pi}\int_{|\zeta|=1} \operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta}$$

$$-\frac{1}{\pi}\int_{|\zeta|<1}\left\{\frac{w_{\bar\zeta}(\zeta)}{\zeta - z} + \frac{z\overline{w_{\bar\zeta}(\zeta)}}{1 - z\bar\zeta}\right\}d\xi d\eta, \qquad |z| < 1.$$

Main Problems

Main problems

Schwarz Problem for Bounded Sectors:

Find a function $w \in C^1(\Omega;\mathbb{C}) \cap C(\overline{\Omega};\mathbb{C})$ *satisfying the following conditions:*

$$\begin{cases} w_{\bar z} = f \ \text{ in } \Omega, \qquad Re\,w = \gamma \text{ on } \partial\Omega, \\ \dfrac{\alpha}{\pi}\displaystyle\int_0^{\pi/\alpha} Im\,w(e^{i\varphi})\,d\varphi = c, \quad c \in \mathbb{R} \end{cases}$$

With $\Omega = \left\{|z| < 1,\ \ 0 < \arg z < \frac{\pi}{\alpha}\right\}$, and

$f \in L_p(\Omega;\mathbb{C}), \qquad p > 2, \qquad \gamma \in C(\partial\Omega;\mathbb{R}), c \in \mathbb{R}$ *is given.*

Main Problems (continued)

Schwarz Problem for Unbounded Sectors:

Find a function $w \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$ *satisfying the following conditions:*

$$\begin{cases} w_{\bar z}(z) = f(z)\,, & z \in \Omega, \\ Re\,w(\zeta) = \gamma(\zeta), & \zeta \in \partial\Omega, \\ Im\,w\left(i^{\frac{1}{\alpha}}\right) = c, & c \in \mathbb{R}. \end{cases}$$

Where $\Omega = \left\{z \in \mathbb{C}:\ \ 0 < \arg z < \frac{\pi}{\alpha}\right\}$, *and*

$f \in L_{p,2}(\Omega;\mathbb{C}), \qquad p > 2$ *and* $\gamma \in L_2(\partial\Omega;\mathbb{R}) \cap C(\partial\Omega;\mathbb{R}), c \in \mathbb{R}$ *is given.*

Results

Theorem: *The Schwarz problem for the bounded sector Ω is uniquely solvable by*

$$w(z) = \frac{\alpha}{2\pi i}\int_{\Gamma_0}\gamma(\zeta)\left(\frac{\zeta^\alpha + z^\alpha}{\zeta^\alpha - z^\alpha} - \frac{\overline{\zeta^\alpha} + z^\alpha}{\overline{\zeta^\alpha} - z^\alpha}\right)\frac{d\zeta}{\zeta} + ic$$

$$+\frac{\alpha}{\pi i}\int_{[\omega,0]\cup[0,1]}\gamma(\zeta)\left[\frac{1}{\zeta^\alpha - z^\alpha} - \frac{z^\alpha}{1 - z^\alpha\zeta^\alpha}\right]\zeta^{\alpha-1}d\zeta$$

$$-\frac{\alpha}{\pi}\int_\Omega\Big[f(\zeta)\left(\frac{1}{\zeta^\alpha - z^\alpha} - \frac{z^\alpha}{1 - z^\alpha\zeta^\alpha}\right)\zeta^{\alpha-1}$$

$$-\overline{f(\zeta)}\left(\frac{1}{\overline{\zeta^\alpha} - z^\alpha} - \frac{z^\alpha}{1 - z^\alpha\overline{\zeta^\alpha}}\right)\overline{\zeta^{\alpha-1}}\Big]d\xi d\eta, \quad z \in \Omega$$

Theorem: *The Schwarz problem for the unbounded sector Ω is uniquely solvable by*

$$w(z) = i\,c + \frac{\alpha}{\pi i}\int_0^{+\infty}\gamma(\zeta)\left(\frac{1}{\zeta^\alpha - z^\alpha} - \frac{\zeta^\alpha}{\zeta^{2\alpha} + 1}\right)\zeta^{\alpha-1}d\zeta$$

$$-\frac{\alpha}{\pi i}\int_0^{+\infty}\gamma((-1)^{1/\alpha}\zeta)\left(\frac{1}{\zeta^\alpha + z^\alpha} - \frac{\zeta^\alpha}{\zeta^{2\alpha} + 1}\right)\zeta^{\alpha-1}d\zeta$$

$$-\frac{\alpha}{\pi}\int_\Omega\Big\{\zeta^{\alpha-1}f(\zeta)\left(\frac{1}{\zeta^\alpha - z^\alpha} - \frac{\zeta^\alpha}{\zeta^{2\alpha} + 1}\right)$$

$$-\bar\zeta^{\alpha-1}\overline{f(\zeta)}\left(\frac{1}{\bar\zeta^\alpha - z^\alpha} - \frac{\bar\zeta^\alpha}{\bar\zeta^{2\alpha} + 1}\right)\Big\}d\xi d\eta.$$

Conclusions

Two Schwarz-type boundary value problems are explicitly solved in two sector domains. The result obtained in this article generalize many recent result citied in the references below.

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